THE ONTOLOGICAL STATUS OF THE PRINCIPLE OF THE EXCLUDED MIDDLE

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ABSTRACT

In this article we want to investigate the ontological status of the logical principle of the excluded middle. In doing this, we are not only taking notice of the notion of "epistemic values" in contemporary philosophy of science since we are also considering issues from the foundational notion of continuity – along with the whole-part relation implied by the latter. The status of the said principle crucially depends on the interconnections between number, space and the meaning of analysis.

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1. Epistemic values and the nature of analysis

Analysis comprises both an act of identification and an act of distinguishing. In Greek philosophy it was realized that these acts of identification and distinguishing are subject to the logical principles of identity and non-contradiction. Already Aristotle framed the principle of the excluded middle (Metaph.1057a33) which claims that any statement is either true or false (cp. Copi, 1978:306).

Since the rise of the Baden school of neo-Kantian thought (Windelband/Rickert) the philosophical legacy of the West has become accustomed to speak about values in stead of principles (or: norms). In particular, the application of rules is distinguished from evaluating (cf. Kuhn, 1977:331; Kuhn, 1984:379). Acknowledging that they are not exhaustive, Kuhn mentions the following five values influencing theory choice: "accuracy, consistency, scope, simplicity, and fruitfulness" (1984:373). In his discussion of epistemic values, McMullin follows Kuhn and also discusses the choice of a theory in terms of value-judgements which differ from the mere application of a rule (cf. 1983:11). He preferably speaks about "epistemic values" and transforms the values mentioned by Kuhn by referring to them as predictive accuracy, internal coherence, external consistency, unifying power and epistemic fertility. To this list epistemic simplicity is added (1983:15-16).

It is clear that this way of dealing with "epistemic values" is dependent on underlying coherences between the analytical facet of theory formation and diverse non-analytic aspects of our experience. For example, "fertility" first of all reminds us of a biotic phenomenon.[1] Plants need "fertile soil" in order to grow properly and bear fruits. Analogously, theories may turn out to be "fruitful" by bearing "fruits". We may refer to this state of affairs by saying that we encounter, in the value of *epistemic fertility*, a *biotical analogy* within the structure of theoretical thought, i.e. within the structure of (deepened) analysis.

Similarly, we may relate *consistency* to the logical principle of *non-contradiction*, which analogously reflect the meaning of *being distinct*, i.e. of the arithmetical meaning of a *discrete quantity*. Number shows its nuclear meaning by way of being a (distinct) multiplicity – first of all evinced in the *succession* of *natural numbers*. Although logicism tries to deny the original arithmetical meaning of quantity by attempting to reduce it to a supposedly *logical* concept of classes, it does not escape from a *petitio principii*. Russell explains his logicistic aim in terms of his attempted logical "definition" of the number 2:

> "1+1 is the number of a class *w* which is the logical sum of two classes *u* and *v* which have no common term and have each only one term. The chief point to be observed is, that logical addition of numbers is the fundamental notion, while arithmetical addition of numbers is wholly subsequent" (1956:119).

1 I am using the unfamiliar term biotic in stead of biological. Strictly speaking, bio-logy} is the scientific study of biotic phenomena. Similar abuses of this suffix is found in terms like psycho-logical (in stead of psychical), socio-logical (in stead of social), and onto-logical (in stead of ontical). Strictly seen, therefore, we should speak about the ontical status of the principle of the excluded middle.

We only have to consider his words "two classes *u* and *v*" to realize the circularity: where does he get hold of the notion "*two*"?[1] He simply uses the number "2" in its original *arithmetical* meaning of *quantity* in order to distinguish the two classes *u* and *v* from each other to start with, and then wants to deduce it again from an act of "logical addition".[2] If our notion of unity and multiplicity first of all relates to arithmetical phenomena, then it is clear that the meaning of the *logical* principle of identity and non-contradiction as such *analogically* reflects this basic arithmetical meaning. Whatever is given as a discrete unity (as being distinct) is *identical* to itself (the basis of the principle of identity) and is *different* from whatever it is not (the basis of the principle of non-contradiction).

2. The twofold nature of an analogy

The notion of an analogy contains two elements: similarity and difference – and these elements are closely connected with the nature of analysis as identifying and distinguishing. To be sure, one can only identify and distinguish on the basis both of similarities and differences. When we differentiate between life in a biotic sense and social life, we are simultaneously confronted with a moment of similarity: the term life, and with a difference: the difference between the "biotical" and the "social". Surely, life in a biotic sense fundamentally differs from life in a social sense.[3] Therefore, *in* this moment of similarity the difference between the social aspect and the biotic aspect reveals itself. It sounds almost paradoxical to say that two aspects show a similarity precisely in that moment which reveals the difference between them.

This kind of a 'difference in terms of similarity' may be called a (modal) analogy.[4] Surely, modal analogies are not the only kind of analogies which one can distinguish, since entities also evince differences in their moments of similarity. Our lingual ability to designate these entitary analogies, as distinct from modal analogies, is known to us in the form of metaphors (viz. "foot of the mountain").[5]

In order to analyze the nature of the logical principle of the excluded middle we only have to consider modal analogies. As a guideline we now have the following two modal analogies: the logical principles of identity and non-contradiction (arithmetical analogies within the modal structure of analysis) and the mentioned value of epistemic fertility (representing a biotical analogy within the modal structure of analysis). From this perspective there is no ground to differentiate between these two values/principles in so far as they both display the same ontical structure – they are examples of an inter-modal coherence between different modal aspects of reality.[6] We want to desig-

1 Already in 1910 Cassirer criticized this circularity: "Die Bestimmung der Zahl durch die Aequivalenz von Klassen setzt voraus, dass diese Klassen selbst als eine Mehrheit gegeben sind" ("The determination of number by means of the equivalence of classes presupposes that these classes themselves are given as a multiplicity") (1969:67-68). In connection with the similarity between classes Cassirer proceeds: "Aber selbst, wenn wir diesem Gesichtspunkt gemäss darauf verzichten könnten, die Einzelklassen, der wir mit einander vergleichen, zuvor in sich selbst numerisch zu gliedern, so bleibe doch stets der Umstand zurück, dass wir die Inbegriffe als Ganzes einander entgegensetzen und sie eben damit als `zwei' verschiedene auffassen müssen" ["But even if we, according to this perspective, beforehand could have left aside the attempt numerically to differentiate the classes we are comparing with each other, then it is still presupposed that we have opposed the classes as wholes, and precisely by doing that should have conceived them as `two' distinct (classes)]" (1969:68).

2 Singh also points out that Russell's attempt makes him a victim of the "vicious circle principle" (1985:76).

3 Cf. Hart who uses the biotical analogy within the structure of the economical aspect to explain a similar state of affairs - 1984:158-159.

4 The term "modal" indicates the various "modes of experience", i.e. the different functional ways in which we are able to experience reality. Traditionally this dimension of reality is related to properties which are different from entities ("substances"). Being alive and being social are two functional properties of human beings. Consequently we can also call modal analogies "property analogies".

5 In passing we note that a metaphor demonstrates two important things: (a) the difference between the logical mode and the sign-mode (or: the difference between concept and word), and (b) the foundational role of the logical mode in relation to the sign-mode. Simply think about the nature of a "boxing ring". If the sign-mode of reality coincides with the logical mode, i.e. if it is not distinct from the logical mode, then this metaphor would be an assertion that a "square circle" exists! (Cassirer refers to this well-known example of an illogical concept - stemming from Russell - in an inverted form: a "round quadrangle" ("rundes Viereck" - 1969:16).

6 An important part of the philosophy of Dooyeweerd is dedicated to an analysis of the inter-modal coherence between the different aspects of reality. Although he does not relate the principles of identity and non-contradiction to the arithmetical analogy present in identifying and distinguishing, he does explain the } {principium rationis sufficientis} in terms of the physical analogy within the modal structure of the analytical aspect (1969:119). Stafleu gives a related description of the nature of scientific prediction and explanation: "Prediction is the first and most obvious aim of any theory. This is a consequence of the

nate all analogies from other modal aspects revealing themselves within the logical-analytical mode in this way as modal logical *principles*. Only when these principles are given a positive form, i.e. when they are "positivized", do we meet logical rules that could be applied, such as in the case of the logical rules of classical Aristotelian predicate logic or those of modern symbolic logic.[1]

3. The universal applicability of the principle of the excluded middle in question

It is known that Brouwer rejects the universal applicability of the principle of the excluded middle (partly) because his neo-intuitionism identifies mathematical existence completely with what could be constructed. For this reason Brouwer states that the question concerning the validity of the principle of the excluded middle is equivalent to the question concerning the possibility of solving mathematical problems (1919:9). He says that whether, in the decimal expansion of π, infinitely many combinations of equal successive digits appear, must be viewed as uncertain (1919:9). Heyting specifically uses examples concerning the occurrence of the sequence 0123456789 in the decimal expansion of π. Write for instance the decimal expansion of π down:

$$\pi = 3.1415 \ldots$$

and the decimal fraction

$$\rho = 0.3333 \ldots$$

which brakes off as soon as the sequence 0123456789 occurs in the decimal expansion of π.[2] By accepting the principle of the excluded middle, the following must be correct:

$$\rho = 1/3 \text{ v } \rho \neq 1/3$$

According to the logic of intuitionism, the expression "$U \vee (\neg U)$" implies that we must be able to construct a proof for every mathematical statement U, or construct, by starting with the assumption that U is valid, a contradiction. But then, the same requirements must apply to the above mentioned case. That is, however, impossible, for to prove one of the statements ($\rho = 1/3$ or $\rho \neq 1/3$), we must first of all be able to decide if the sequence 0123456789 does occur in the decimal expansion of π. Since our present state of mathematical knowledge does not allow this, intuitionism rejects the universal scope of the principle of the excluded middle – whenever the infinite is at stake it is inapplicable.

With his arguments Brouwer convinced Ludwig Wittgenstein and Hermann Weyl. Wittgenstein explains his stance in terms of a similar example. It seems self-evident that the sequence 7777 either occurs somewhere in the decimal expansion of π, or it does not:

> "We want, that is, to quote the law of the excluded middle and to say: `Either such an image is in his mind, or it is not; there is no third possibility!' – We encounter this queer argument also in other regions of philosophy. `In the decimal expansion of π either the group 7777 occurs, or it does not – there is no third possibility'. That is to say `God sees – but we don't know'. But what does that mean? – We use a picture; the picture of a visible series which one person sees the whole of and another not. The law of the excluded middle says here: It must either look like this, or like that. So it really – and this is a truism – says nothing at all, but gives us a picture" [1968:112 (par.352); cp. p.127 (par.426)].

These examples are dependent on an underlying divergence which stand in relation to the opposing evaluation of the nature of the infinite by intuitionism and the other trends in modern mathematics.

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deductive character of a theory, i.e., its kinematic foundation, deduction being the logical movement from one statement to another. We shall characterize prediction to be the `kinematic' function of a theory, to be distinguished from its `physical' function, which is to explain. Explanation is tied to a cause-effect relation of some kind" (1987:31).

1 The distinction between modal logical norms/principles/values and their concretization/positivization rests on the distinction between the determining and delimiting norm-side of the logical aspect and that which is correlatively factually subjected to this determining norm-side. Concepts, statements and arguments are factually subjected to the logical principles/norms. Only when we acknowledge this distinction is it possible to differentiate between logically correct concept formation, statements and arguments and illogical concept formation, statements and arguments.

2 As yet it is completely unknown if this sequence (i.e., 0123456789) does occur in the decimal expansion of p and there is know method known to determine its existence.

Traditionally a distinction is drawn between the potential infinite (PI) and the actual infinite (AI). With reference to Aristotle (cf. Phys.208a6), Cantor distinguishes between *apeiron dunamei* (*ἀπειρόν δυναμει*) and *apeiron hos aforismé*non (*ἀπειρόν ὁ ἀωρισμένον*) (the PI and the AI). His general description is:

> "We preferably refer to the PI where we encounter an undetermined varying finite magnitude that either increases beyond all finite limits ...... or decreases beneath all finite limits of smallness ...; generally I always speak of PI when we consider an undetermined magnitude which allows for a numberless multiplicity of determinations" (1962:401).[1] "The AI, on the other hand, is to be seen as a quantity (Quantum) which is not varying, since it is firm and determined in all its parts, a genuine constant, although at the same time it surpasses every similar finite magnitude in size" (1962:401).

Cantor also refers to the PI as the "uneigentliches Unendliches" ("improper infinite"), since its reality is dependent on the AI which alone makes it possible (1962:404).

In radical opposition to Cantor's employment of the AI, Weyl supports Brouwer's rejection of the AI:

> "Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the `absolute' that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence" (1946:9).

3.1. The whole-part relation

The relevance of the distinction between the PI and AI for the principle of the excluded middle is first of all seen in relation to the notion of a totality or a whole. Cantor speaks about a "Quantum" which is "firm and determined in all its parts" (1962:401), thus implying the nature of a whole or totality. When "all the parts" are present, a whole is totally given (cf. Foradori, 1933) – and only when we have a whole is it possible to execute a division entailing an either/or. Clearly, therefore, if an infinite totality (the AI) is rejected by intuitionism, then also the applicability of the principle of the excluded middle in the case of the infinite cannot be upheld.[2]

Whereas it is not difficult to locate the most primitive meaning of infinity as belonging to our arithmetical intuition of one, another one, and so on without an end, i.e. endlessly, infinitely,[3] it is not so clear why we have to distinguish this primitive meaning of infinity (i.e. endlessness) from the meaning of the actual infinite. Is there no logical continuity between the acceptance of an endless sequence and the nature of the actual infinity of this sequence? Those committed merely to the acknowledgement of the potential infinite do not allow for any fashion of presenting the multiplicity of elements of such an endless sequence as being given at once, without any succession. Kaufmann, for example, explicitly speaks about the unfounded supposition of "the actual infinite as a totality of discrete elements" (1968:144).

At this stage it must be clear that our problem now concerns the question whether it is possible to reduce the notion of a totality (whole) to the quantitative meaning of number. Since Greek philosophy and mathematics indeed provides us with a deep insight into the nature of the whole-part relation we may allow ourselves a succinct digression dealing with this heritage.

3.2. Starting-points in Greek philosophy

After the discovery of incommensurability by the Pythagorean thinker Hippasos of Metapontum (cf. Von Fritz, 1965), the school of Parmenides gave up the aim to see everything as number. Consequently, the subsequent Eleatic school characterizes their new metaphysics of being in terms derived from our intuition of space. As hallmarks of being the following are mentioned: "since it is

1 Note that Cantor still uses the notion of a magnitude (a term with a geometrical descent), in stead of (numerical) value.

2 Intuitionism prefers not to speak about a set in the traditional sense of the word. When a common mode of generation for its elements is defined it introduces the notion of a }{spread}{\plain \fs24 , and when a characteristic property of its elements is meant the term }{species}is used (Heyting, 1971:36). Of course, according to intuitionism, the law of the excluded middle does hold in the case of }{finite}totalities.

3 In his chapter dealing with the "arithmetization of mathematics", Voss refers to Husserl and Wundt in connection with the psychological notion of successive thought acts (1913:33). It was W.R. Hamilton who, in a typical Kantian fashion, defined algebra as the "science of pure time" (1833). We shall argue below that the notion of time has a place in mathematics without understanding it in a psychological sense.

unborn it is imperishable,[1] .... it was not and will never be because it is contiguously given in the present as an indivisible whole, unified, coherent" (B Fragment 8, 3-6). Note the subtle emphasis on the present (Greek: νυν), which provided the starting-point for the Western speculation about eternity as the timeless present (cf. Plotinus, Enneade III,7 and Beierwaltes, 1967).[2] These perspectives are further developed by Parmenides in B Fragment 8, 22-25, where discontinuity is negated: being closes upon being.

However, especially the Eleatic philosopher Zeno, for the first time, explores this (spatial) whole-part relation further. Let us look at his B Fragment 3:

> If things are a multiplicity, then it is necessary that their number must be identical to their actual multiplicity, neither more nor less. But if there are just as many as there are, then their number must be limited (finite). If things are a multiplicity, then necessarily they are infinite in number; for in that case between any two individual things there will always be other things and so on. Therefore, then, their number is infinite.

Although the two main parts of this fragment starts with the assumption that things are many, opposite conclusions are reached. Since the terms used by Zeno originates from the modal meaning of the spatial aspect (even though not understood by him as such), it is not at all far-fetched to suppose that the two sides of the spatial whole-part relation underlie this argument. If the multiplicity of the first section refers to the many parts of the world as a whole, it stands to reason that taken together they constitute the unity of the world as a whole (and that their number would be limited). If, on the other hand, one starts with the whole and then tries to account for its parts, one must keep in mind that between any two of the many parts there will always be other, indicating an infinity of them. Fränkel explicitly uses the whole-part relation to explain the meaning of this fragment (1968:425 ff., 430).

If this interpretation is sound, then Zeno not only understood something of the whole-part relation, but also for the first time realized that spatial continuity is characterized by being infinitely divisible. This perspective can also be used in support of the interpretation given to Zeno's B Fragment 1 by Hasse and Scholz (1928:10-13). This first Fragment (which we inherited from Simplicius) states that if there exists a multiplicity, then simultaneously it must be large and small; large up to infinity and small up to nothingness. Hasse and Scholz clarify this fragment by interpreting it as follows: If it is permissible to conceptualize a line-stretch as an aggregate of infinitely many small line stretches, then there are two and only two possibilities. Every basic line segment either has a finite size (larger than zero), in which case the aggregate of line-stretches transcends every finite line-stretch; or the supposed line-stretches are zero-stretches in the strict sense of the word, in which case the composed line is also a zero-stretch, because the combination of zero-stretches can always only produce a zero-stretch, however large the number of zero-stretches used may be (1928:11).

Besides the fact that we can render the two mentioned fragments of Zeno perfectly intelligible by using the spatial whole-part relation, further support for this understanding may also be drawn from the account which Aristotle gives of Zeno's arguments (cf. Metaph.233a13 ff. and 239b5 ff.). One of the standard expositions of Zeno's argumentation against the reality of motion is completely dependent on the employment of the spatial whole-part relation with its implied trait of infinite divisibility. Guthrie explains this argument by saying that according to Zeno "Motion is impossible because an object moving between any two points A and B must always cover half the distance before it gets to the end. But before covering half the distance it must cover half of the half, and so on ad infinitum. Thus to traverse any distance at all it must cover an infinite number of points, which is impossible in any finite time" (1980:91-92).

1 Cp. B Fragments 2 and 3 of Anaximander contained in Diels-Kranz, 1959-1960.
2 Wittgenstein still echoes this legacy in his Tractatus 6.4311: "If we take eternity to mean not infinite temporal duration but timelessness, then eternal life belongs to those who live in the present" (1966:147).

This spatial orientation explains why subsequent Greek thinkers explored still further the spatial whole-part relation. Anaxagoras claims that we are not entitled to speak about the smallest, since there always exists something smaller. That what is can never cease to be through continued division, no matter how far this process of division is carried through (B Fragment 3). And since no smallest can exist, it is impossible (for any part) to be separated and set on its own, because it must now as in the beginning, exist together with everything else (B Fragment 6). This "existence together/coherent existence" refers to the coherence of spatial continuity in which all (material) things are fitted. However, this continuity is not composed out of discrete (separated) parts, as if they were cut apart by an axe (B Fragment 8).

3.3. Irreducibility of the totality-character of continuity

With these distinctions Anaxagoras not only anticipates the view of Aristotle, since he also provides modern intuitionistic mathematics with valuable insights. Weyl is most explicit about this heritage: "Yes, especially now, in the foundations of mathematics, we are everywhere invited immediately to go back to the Greeks".[1] When he discusses the "present" state of mathematical knowledge in 1926 he starts with Anaxagoras (1926:1-2). He often characterizes continuity in terms of the whole-part relation (cf. 1921:77, and 1966:74).

Intuitionistic mathematics here also follows fundamental insights from Aristotle. Although Aristotle analyzed both space and number within the perspective of one category, namely that of quantity (number is a discrete quantity and space is a continuous quantity), he nonetheless develops a remarkable insight into the structure of spatial continuity. The parts of a discrete quantity possess no common boundary, whereas, in the case of a line (as a continuous quantity) it is always possible to detect a common limit of its parts (Categoriae, 4b25 ff., 5a1 ff.): "The act in which a continuous distance is divided into two halves takes one point twice since it is viewed as starting-point and end-point" (Physica, 263a23 ff.).[2] According to Aristotle it is therefore self-evident that "everything continuous is divisible into divisible parts which are infinitely divisible" (Physica, 231b15 ff.).

It is especially this trait of continuity that is taken seriously in intuitionistic mathematics. Weyl points out: "That is has parts, is a basic property of the continuum", and adds: "... it belongs to the very essence of the continuum that every one of its parts admits a limitless divisibility" (1921:77).

According to Weyl, the general aim of Weierstrass, Dedekind and Cantor, namely to arithmetize spatial continuity completely, had to take recourse to the *neighbourhood* concept: "To account for the continuous coherence of the points, contemporary analysis, which has separated the continuum into a set of isolated points, takes refuge to the neighbourhood concept" (1921:77).

However, it is not at all imperative to adhere to the intuitionistic approach in modern mathematics in order to realize that the totality-character of continuity is irreducible to numerical notions. Bernays did sense the irreducibility of the spatial whole-part relation (the totality-property of spatial continuity) with an astonishing lucidity: "The property of being a totality "undeniably belongs to the geometric idea of the continuum. And it is this characteristic which resists a complete arithmetization of the continuum".[3]

In another context he even states that the classical foundation of the real numbers given by Cantor and Dedekind does not "manifest a complete arithmetization" (1976:187-188). To this he adds the remark: "It is in any case doubtful whether a complete arithmetization of the idea of the continuum could be justified. The idea of the continuum is any way originally a geometrical idea" (1976:188).

1 "Ja gerade heute sehen wir uns genötigt, überall in den Grundlagen der Mathematik wieder unmittelbar auf den Griechen zurückzugehen" (1931:1).

2 Cf. Foradori, 1933:162,166. Böhme highlights the striking similarities between the Aristotelian conception of the continuum and the Cantor-Dedekind characterization thereof. He shows that although the latter employs the actual infinite – totally rejected by Aristotle – they still conform to the two criteria which Aristotle developed for continuity (1966:308 ff.).

3 "Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" – 1976:74).

His deeply felt reaction against the mistaken and one-sided nature of modern arithmeticism is best seen from his following remark:

> "The arithmetizing monism in mathematics is an arbitrary thesis. The claim that the field of investigation of mathematics purely emerges from the representation of number is not at all shown. Much rather, it is presumably the case that concepts such as a continuous curve and an area, and in particular the concepts used in topology, are not reducible to notions of number (Zahlvorstellungen)" (1976:188).

4. A crucial inter-modal coherence

We may summarize the outcome of our preceding considerations by saying that the fundamental difference between the potential infinite and the actual infinite stands and falls with the difference between the primitive meaning of an arithmetical succession (preferably designated as the successive infinite), and the totality-character of an actual infinite set (where all the elements are viewed as being present simultaneously). In the latter case we prefer to speak about the at once infinite.[1]

It is also clear from Cantor's notion of a set that he explicitly uses the feature of being a totality to characterize it: "We understand a `set' to be any collection into a whole M of definite and distinct objects m of our intuition or our thought (which are called the `elements' of M)".[2] Since the notion of a *set* and that of an *element* are indefinable terms in Zermelo-Fraenkel set theory, the formulation of the "Axiom of Infinity" gives the impression that only the successive infinite is intended (cf. Fraenkel et al., 1973:46). In particular, when the "Axiom of Infinity" is read in coherence with the "Axiom of Power-Set" (cf. Fraenkel et al., 1973:35), it is clear that together they (also) cover the case of infinite totalities (where all the subsets of an infinite set also consitute a new set).[3]

It must be clear that our argument does not introduce continuity in its irreducible spatial sense into the meaning of number. Much rather, we here need the above mentioned figure of a *modal analogy*: any arithmetical succession may, under the guidance of our spatial intuition of simultaneity, be viewed *as if* all its elements are present *at once* – in which case we have employed the regulative idea of the *at once infinite*, accounted for in terms of an anticipatory analogy of space within the meaning of number. In other words, the use of the at once infinite (with its irreducibility to the successive infinite) stands and falls with the irreducibility of the spatial time-order of simultaneity which is simply the determining order for any coherent totality, since to constitute any spatial whole all the parts have to be present at once.[4] In the development of Greek philosophy and mathematics it was conjectured to view our intuition of space as being more basic than that of number. Fraenkel et al. remark: "Certainly the discrete admits an easier access to logical analysis, and the tendency of arithmetization, already underlying Zeno's paradoxes, has been impressing its mark upon modern mathematics and may be perceived in axiomatics of set theory as well as in metamathematics. However, the converse direction is also conceivable, for intuition seems to comprehend the continuum *at once* (I am emphasizing – DS); mainly for this reason Greek mathe-

1 These terms were already used in the disputes of the early 14th century about the infinity of God. Compare the expressions *infinitum successivum*and *infinitum simultaneum* (Maier, 1964:77-79).

2 "Unter einer `Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die `Elemente' von M genannt werden) zu einem Ganzen" (1962:282). In his description of the nature of "Teilmengen" ("subsets") on the same page, the word "at once" ("zugleich") is used constitutively. A good discussion of Cantor's concept of a set is found in Singh (1985).

3 We note in passing that it was the postulation of an infinite `class' which showed the untenability of Russell's logicistic project. In 1919 he had to admit that all of his earlier proofs for the existence of an infinite class are invalid (1919:134-135; cf. Morris, 1929:456). Fraenkel et al., remark: "It seems, then, that the only really serious drawback in the Frege-Russell thesis is the doubtful status of InfAx, according to the interpretation intended by them" (1973:186).

4 Also in the aspects of number and space we have to distinguish between a law-side and a factual side. The former plays a limiting and determining role, whereas the latter is limited and determined by the order at the law-side. The history of time-measurement suggests that time should be seen as a unique dimension of reality which cannot be identified with the physical aspect of reality. First of all time-measurement used the numerical time-order of succession – by counting the days, the weeks, months and years. Then they used the spatial time-order of simultaneity – with the aid of instruments like sundials. Subsequently, the kinematical order of constancy was used (cp. the regular swing of a pendulum in a mechanical clockwork). Finally, in our century, we are using the irreversible physical time-order – for example in atomic clocks and in the procedure used to determine the age of the different earth layers. This perspective continues insights of Descartes - cf. his notion of *numeros*, *ordo*and *duratio* (cf. Becker 1973:269) – and Kant – cf. his three modes of time namely succession, co-existence and duration (1956:B-219). The acknowledgement of time as a unique dimension of reality is also worked out by Stafleu in his analysis of the foundations of physics (cf. Stafleu, 1980:16, 83 ff.).

matics and philosophy were inclined to consider continuity to be the simpler concept and to contemplate combinatorial concepts and facts from an analytic view" (1973:213).

Although we want to acknowledge the spatial descent of the notion of a whole and its parts,[1] it must also be clear that the meaning of spatial continuity presupposes that of number. Consequently, where the infinite divisibility of any spatial continuum refers back (retrocipates) to the primitive meaning of the successive infinite, the at once infinite refers forward (i.e. anticipates) to the spatial order of simultaneity with its factual correlate, the totality-character implied and determined by this order of at once.

5. The principle of the excluded middle: a retrocipation to an anticipation

Against the foregoing background we may now conclude by formulating the ontical status of the principle of the excluded middle. As such it is first of all a part of the arithmetical analogy within the modal structure of the logical-analytical mode, intimately connected with the principles of identity and non-contradiction. To be sure, in the finite case, the bifurcation of A and non-A clearly excludes any third possibility.

However, in order to ensure the universal applicability of this logical principle, i.e. also in the case of the infinite, we have to acknowledge the (irreducible) meaning of the at once infinite, which itself is completely dependent on the irreducibility of the spatial order of simultaneity with its implied correlate: the whole-part relation. Only under the anticipatory guidance of the regulative hypothesis of the at once infinite are we justified in accepting that the principle of the excluded middle holds in the infinite case as well. Therefore, via the (retrocipatory) analogy of number within the structure of analysis, this principle finds its ultimate foundation in the numerical anticipation to the meaning of space, which entitle us to say that the ontical status of the principle of the excluded middle is given in its being a retrocipation to an anticipation![2]

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1 In the light of this perspective the following remark of Bernays is quite understandable: "The idea of the continuum is a geometrical idea which analysis expresses in terms of arithmetic" ("Die Idee des Kontinuums ist eine geometrische Idee, welche durch die Analysis in arithmetischer Sprache ausgedrükt wird" – 1976:74). Further on this page, in connection with shortcomings in the intuitionistic conception, he adds that it is precisely this *totality-feature* of the continuum which resists arithmetization: "This stems from the fact that on the intuitionistic conception, the continuum does not have the character of a totality, which undeniably belongs to the geometrical idea of the continuum. And it is this characteristic of the continuum which would resist perfect arithmetization" ("Das rührt davon her, dass die intuitionistische Vorstellung nicht jenen Charakter der Geschlossenheit bezitzt, der zweifellos zur geometrischen Vorstellung des Kontinuums gehört. Und es ist auch dieser Charakter, der einer vollkommenen Arithmetisierung des Kontinuums entgegensteht" – 1976:74).

2 In other words, the meaning of the principle of the excluded middle is given in a retrocipation from the logical-analytical mode to the arithmetical mode, which in turn, in the at once infinite, anticipates the factual spatial whole-part relation as subjected to and as determined by the spatial time-order of simultaneity.

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